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A RIEMANN HYPOTHESIS TRY

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Introduction

The Riemann zeta function is the function of the complex variable s, defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series $\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$ and in the whole complex plane C (except at s=1) by analytic continuation. The zeros of this function are the solutions $\rho \in \mathbb{C}$ of the equation $\zeta(\rho) = 0$. The attention of the present research on the Riemann zeta function deals with the zeros with real part in $]0;1[$. These last zeros are called nontrivial zeros for certain reasons you can find in the official problem description. The Riemann hypothesis that has to be proved is that all the nontrivial zeros have real part equal to $\frac{1}{2}$.

A Riemann hypothesis confirmation?

s and s' are two nontrivial zeros of the analytic continuation on $\mathbb{C} - \{1\}$ using the η function of the Riemann ζ function, using "s", $\zeta(s) = \frac{1}{1-2^{1-s}}\eta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{+\infty} (-1)^{n+1} n^{-s}$.

$$s = a + ib,\ 0 < a < 1,\ b \in \mathbb{R}$$
$$s' = a' + ib',\ 0 < a' < 1,\ b' \in \mathbb{R}$$

$\zeta(s) = 0 \Rightarrow \eta(s) = 0,\ \zeta(s') = 0 \Rightarrow \eta(s') = 0$

$\Rightarrow \Re(\eta(s)) = \Re(\eta(s')) = 0 \Rightarrow \sum_{n=1}^{+\infty} (-1)^{n+1} n^{-a} \cos(b\ln(n)) = \sum_{n=1}^{+\infty} (-1)^{n+1} n^{-a'} \cos(b'\ln(n)) = 0$

If we apply the Taylor's theorem to the function $b \to \cos(b\ln(n))$, for any n in $\mathbb{N} - \{0\}$, we have $\cos(b\ln(n)) = 1 + o_{b\to 0}(b)$.

So, we have $(1 + o_{b\to 0}(b)) \sum_{n=1}^{+\infty} (-1)^{n+1} n^{-a} = 0$, which implies $\sum_{n=1}^{+\infty} (-1)^{n+1} n^{-a} = 0$.

In the same way, we have $\sum_{n=1}^{+\infty} (-1)^{n+1} n^{-a'} = 0$.

We make a subtraction of the two series and have: $\sum_{n=1}^{+\infty} (-1)^{n+1} (n^{-a} - n^{-a'}) = 0$. This new series converges, so the sequence $\left((n^{-a} - n^{-a'})\right)_{n\in\mathbb{N}-\{0\}}$ must converge to zero if $n \to +\infty((-1)^{n+1} \neq 0)$.

The Taylor's theorem gives $n^{-a} - n^{-a'} = (a - a')\ln(n)) + o_{a\to 0}(a) + o_{a'\to 0}(a')$. This sequence converges to zero if $n \to +\infty$. We divide by $\ln(n)$,which does not

change the limit of the sequence $((a-a')\ln(n)) + o_{a\to 0}(a) + o_{a'\to 0}(a'))_{n\in\mathbb{N}-\{0;1\}}$, because $\lim\limits_{n\to+\infty} \frac{1}{\ln(n)} = 0$, and we have $\lim\limits_{n\to+\infty} (a-a') + \frac{o_{a\to 0}(a) + o_{a'\to 0}(a')}{\ln(n)} = 0$.
$\lim\limits_{n\to+\infty} \frac{o_{a\to 0}(a) + o_{a'\to 0}(a')}{\ln(n)} = 0$, so $a = a'$.

Hardy proved[1] there was an infinity of nontrivial zeros with $\frac{1}{2}$ as real part, and we just proved that two nontrivial zeros have the same real part. All nontrivial zeros have $\frac{1}{2}$ as real part.

References

1.G.H. Hardy. "Sur les zéros de la fonction $\zeta(s)$ de Riemann". French. In: *Comptes Rendus de l'Académie des sciences*, Volume 158, 1914, pp.1012-14. ISSN: 00014036.

Abstract. We may confirm the Riemann hypothesis.